Pergamon

www.elsevier.com/locate/jappmathmech

PII: S0021-8928(02)00021-7

J. Appl. Maths Mechs, Vol. 66, No. 2, pp. 165–170, 2002

0021-8928/02/$—see front matter

THE EQUILIBRIUM CONDITIONS OF A ROD ON A ROUGH PLANE†

A. S. SMYSHLYAYEV and F. L. CHERNOUS'KO

Moscow

(*Received* 22 *June* 2001)

The problem of the equilibrium of a rod on a rough horizontal plane when there are dry friction forces is considered. The equilibrium conditions which ensure that the rod remains at rest are determined by solving the problem of an extremum. The results obtained are compared with the well-known result for a Zhukovskii bench. © 2002 Elsevier Science Ltd. All rights reserved.

1. FORMULATION OF THE PROBLEM

Consider a rigid body, representing a rod, at rest on a rough horizontal plane. We will introduce a Cartesian system of coordinates Oxy so that the x axis is directed along the rod, while the point O coincides with one of its ends (Fig. 1). We will denote the length of the rod by l and the density per unit length by ρ. We will assume that the rod is infinitely thin and that its weight is distributed uniformly over the whole length. This assumption corresponds to the case when the rigidity of the rod about the y axis is considerably less than the rigidity about the vertical axis. Suppose external forces are applied to the rod in the Oxy plane. We will denote the principal vector of these external forces by $\mathbf{F}$, its projections on to the x and y axes by F_x and F_y respectively, and the principal moment of all the external forces about the centre of the rod by M_0. Dry friction forces, which obey Coulomb's law, act between the rod and the plane. We will write Coulomb's law in the form

$$\sqrt{X^2+Y^2} \leqslant \rho g k, \quad x \in [0, l] \tag{1.1}$$

where k is the friction coefficient, and $X(x)$ and $Y(x)$ are the linear densities of the projections of the friction forces on to the x and y axes respectively.

We will write the equilibrium conditions of the rod acted upon by the applied forces as

$$F_x + \int_0^l X dx = 0, \quad F_y + \int_0^l Y dx = 0, \quad M_0 + \int_0^l \left(x - \frac{l}{2}\right) Y dx = 0 \tag{1.2}$$

We formulate the problem as follows: it is required to find the conditions which must be imposed on $\mathbf{F}$ and M_0 so that the rod is in a state of equilibrium. Hence, we must find for what values of F_x, F_y and M_0, functions $X(x)$ and $Y(x)$ exist which satisfy relations (1.1) and (1.2).

2. THE EQUIVALENT PROBLEM OF AN EXTREMUM

To simplify the subsequent calculations we will introduce the dimensionless variables

$$x' = \frac{x}{l}, \quad y' = \frac{y}{l}, \quad X' = \frac{X}{\rho g k}, \quad Y' = \frac{Y}{\rho g k}$$

$$F_x' = -\frac{F_x}{\rho g k l}, \quad F_y' = -\frac{F_y}{\rho g k l}, \quad M' = \frac{M_0}{\rho g k l^2} \tag{2.1}$$

and we will write conditions (1.2) in the form (everywhere henceforth we will omit the prime on the dimensionless variables)

$$\int_0^1 X dx = F_x, \quad \int_0^1 Y dx = F_y, \quad \int_0^1 \left(x - \frac{1}{2}\right) Y dx = M \tag{2.2}$$

†*Prikl. Mat. Mekh.* Vol. 66, No. 2, pp. 177–182, 2002.

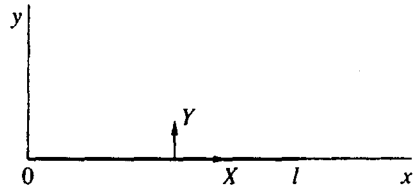

Fig. 1

Condition (1.1) takes the form

$$X^2 + Y^2 \leqslant 1, \quad x \in [0, 1] \tag{2.3}$$

It is required to determine the set Ω in the three-dimensional space (F_x, F_y, M), for the points of which equilibrium occurs, i.e. functions $X(x)$ and $Y(x)$ exist which satisfy relations (2.2) and (2.3).

We will first establish some general properties of the set Ω.

1. The set Ω is convex. This property follows from the convexity of the set of permissible equations, determined by condition (2.3), and from the linearity of transformation (2.2).

2. The set Ω is symmetrical about the three coordinate planes $F_x = 0$, $F_y = 0$ and $M = 0$. Really, if the functions $X(x)$ and $Y(x)$ correspond to the set (F_x, F_y, M), then, as can easily be verified using conditions (2.2) and (2.3), for the set $(-F_x, F_y, M)$ we can take the functions $-X(x)$ and $Y(x)$, for the set $(F_x, -F_y, M)$ we can take the functions $X(x)$ and $-Y(1-x)$, and for the set $(F_x, F_y, -M)$ we can take the functions $X(x)$ and $Y(1-x)$.

We will formulate the problem of the optimum control

$$\begin{gathered} y_1' = X, \quad y_2' = Y, \quad y_3' = (x - 1/2)Y, \quad X^2 + Y^2 \leqslant 1 \\ y_1(0) = y_2(0) = y_3(0) = 0 \\ y_1(1) = F_x, \quad y_2(1) = F_y, \quad y_3(1) \to \max \end{gathered} \tag{2.4}$$

The primes denote derivatives with respect to x, and the quantities X and Y play the role of control functions.

Solving (2.4), we obtain the greatest possible value of M for each pair F_x, F_y. In view of the convexity of the set Ω this means that we will obtain part of the boundary of this set. The remaining parts of the boundary are obtained using the above-mentioned properties of symmetry of this set. Hence, to determine the set Ω it is sufficient to solve the problem of the optimum control (2.4) for different $F_x \geqslant 0$ and $F_y \geqslant 0$.

3. THE SOLUTION OF THE PROBLEM OF THE OPTIMUM CONTROL

The Hamiltonian for problem (2.4) has the form

$$H = p_1 X + p_2 Y + p_3(x - ½)Y \tag{3.1}$$

Here $p_i (i = 1, 2, 3)$ are conjugate variables. If follows from the conjugate system that all the p_i are constant. Without loss of generality we can put $p_3 = 1$.

We will obtain the maximum of Hamiltonian (3.1) under the limitation $X^2 + Y^2 \leqslant 1$. It is obvious that the required maximum is reached when $X^2 + Y^2 = 1$. We will use the method of Lagrange multipliers to determine X and Y, which give a maximum to the Hamiltonian

$$L = p_1 X + p_2 Y + (x - 1/2)Y + \lambda(X^2 + Y^2)$$

Equating the partial derivatives $\partial L/\partial x$ and $\partial L/\partial y$ to zero, we obtain equations, by solving which for X and Y we obtain

$$X = \frac{p_1}{[p_1^2 + (p + x)^2]^{½}}, \quad Y = \frac{p + x}{[p_1^2 + (p + x)^2]^{½}}; \quad p = p_2 - \frac{1}{2} \tag{3.2}$$

Here we have chosen the signs corresponding to the maximum of Hamiltonian (3.1). Since p_1 = const and X does not change in sign when $x \in [0, 1]$, then to satisfy the first condition of (2.2) when $F_x \geqslant 0$, it is necessary that $p_i \geqslant 0$.

Taking the integrals (2.2), we obtain

$$F_x = p_1 \ln \frac{p+1+\Delta_1}{p+\Delta_0}, \quad F_y = \Delta_1 - \Delta_0; \quad M = \frac{1}{2}\Delta_0 - \frac{p_1}{2} F_x - \frac{p}{2} F_y \tag{3.3}$$

$$\Delta_1 = \sqrt{p_1^2 + (p+1)^2}, \quad \Delta_0 = \sqrt{p_1^2 + p^2}$$

Formulae (3.3) define the parametric dependence $M(F_x, F_y)$ in terms of the parameters p_1 and p. It is not possible to obtain this relation in explicit form by eliminating p_1 and p. We will first consider special cases in which an analytical solution is available.

1. Suppose $F_x^2 + F_y^2 = 1$. We will shown that infinitely large values of the conjugate variables correspond to this case. We put

$$p_1 = \mu q_1, \quad p = \mu q_2, \quad \mu \to \infty \tag{3.4}$$

where q_1 and q_2 are new constants and μ is a large parameter. Substituting expression (3.4) into the first two relations of (3.3) and expanding them in inverse powers of μ, we obtain

$$F_x = \frac{q_1}{\sqrt{q_1^2 + q_2^2}}, \quad F_y = \frac{q_2}{\sqrt{q_1^2 + q_2^2}} \tag{3.5}$$

so that the condition $F_x^2 + F_y^2 = 1$ is satisfied. We substitute relations (3.4) into expression (3.2) and put $\mu \to \infty$. Comparing the results obtained with Eqs (3.5), we see that $X \equiv F_x$, $Y \equiv F_y$. Consequently, by the last equation of (2.2) we have $M = 0$. Hence, for all point on the circle $F_x^2 + F_y^2 = 1$ the principal moments of the external forces must equal zero.

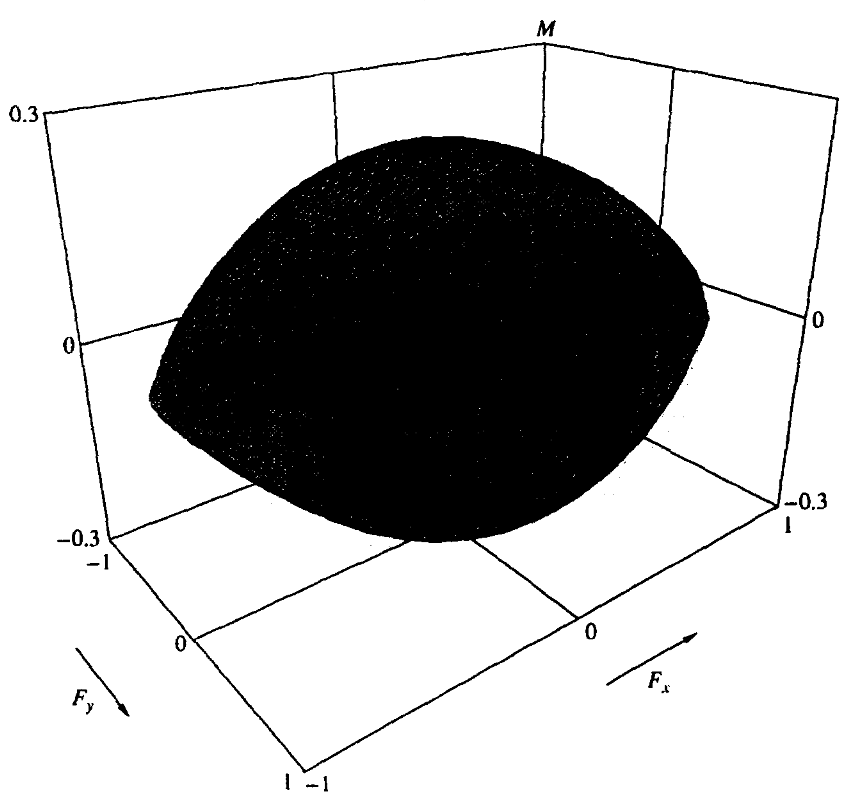

Fig. 2

2. Suppose $F_x = F_y = 0$. Then

$$\int_0^1 X dx = 0$$

and since $X \geqslant 0$ when $x \in [0, 1]$, then $X \equiv 0$. It follows from relations (3.2) that $p_1 = 0$. Substituting this value into (3.3) we obtain $F_y = 0, p = -1/2$ and $M = 1/4$. Hence, the maximum moment when the principal vector of the forces is equal to zero is $1/4$. It is clear from physical considerations that this value of the moment is the maximum possible value for any F_x and F_y.

The system of linear equations (the first two equations of (3.3)) in p_1 and p was solved numerically by Newton's method for each pair F_x, F_y, which satisfy the relation $F_x^2 + F_y^2 \leqslant 1$. Then, we determined the moment M from the third formula of (3.3). The results of the calculation are presented in Fig. 2, where we show the set Ω obtained, in Fig. 3, where for $M \geqslant 0$ we show its sections by the planes $F_x = \text{const}$ (the values of F_x are indicated in the figure), and in Fig. 4, where for $F_y \geqslant 0$ we show sections of the set Ω by the planes $M = \text{const}$ (the values of M are shown in the figure). In view of the symmetry properties of the set Ω about the planes $M = 0$ and $F_y = 0$ we only shown half the corresponding sections in Figs 3 and 4.

4. COMPARISON WITH ZHUKOVSKII'S RESULT

In [1] Zhukovskii obtained the equilibrium conditions of a body resting on two points, acted upon by a normal force applied in the middle of the section connecting these points, and also force **F**, lying in the xy plane. More general equilibrium conditions of bodies on a plane were considered [2] for different conditions of rest (guaranteed equilibrium conditions in the case of static uncertainty). We will denote the angle between the vector **F** and the x axis by φ and the distance from the origin of coordinates to the line of action of the force **F** by $h > 0$ (Fig. 5). The equilibrium conditions, obtained in [1], reduce to the form [2]

$$\frac{F}{Nk} = \begin{cases} [1 + h^2(a\cos\varphi)^{-2}]^{-1/2}, & ha^{-1} < \cos^2\varphi\,|\sin\varphi|^{-1} \\ (ha^{-1} + |\sin\varphi|)^{-1}, & ha^{-1} \geqslant \cos^2\varphi\,|\sin\varphi|^{-1} \end{cases} \tag{4.1}$$

Here a is half the distance between the masses, and N is the normal force applied in the middle of the segment connecting the point masses. In order to compare Zhukovskii's result with the solution of the problem obtained in Section 3, we will change to the dimensionless variables given in (2.1) and put $a = l/2$, $N = 2mg = \rho lg$ and $s = h/a$. We obtain

$$F_x = \xi(\varphi, s)\cos\varphi, \quad F_y = \xi(\varphi, s)\sin\varphi, \quad M = \tfrac{1}{2}s\xi(\varphi, s) \tag{4.2}$$

$$\xi(\varphi, s) = \begin{cases} [1 + (s/\cos\varphi)^2]^{-1/2}, & s < \cos^2\varphi\,|\sin\varphi|^{-1} \\ (s + |\sin\varphi|)^{-1}, & s \geqslant \cos^2\varphi\,|\sin\varphi|^{-1} \end{cases}$$

Equations (4.2), which specify the parametric relation $M(F_x, F_y)$, can be solved for φ and s. We have

$$M = \frac{1}{2} \times \begin{cases} |F_x|(1 + (F_x^2 + F_y^2)^{-1})^{1/2}, & F_y < F_x^2 + F_y^2 \\ (1 - F_y), & F_y \geqslant F_x^2 + F_y^2 \end{cases} \tag{4.3}$$

Formula (4.3) is true when $F_y \geqslant 0$, $-1 \leqslant F_x \leqslant 1$. Relations (4.3) determine the set Ω^* in the space of the quantities (F_x, F_y, M), the points of which correspond to equilibrium states. The set Ω^* is similar to the set Ω and possesses the same symmetry properties.

We will compare the values of the moment given by formula (4.3) with the results obtained in Section 3. It is easy to see that when $F_x^2 + F_y^2 = 1$ the value of the moment $M = 0$, as in the case of a rod. When $F_x = F_y = 0$ the moment reaches its maximum value (for all F_x, F_y) and is equal to $1/2$, which is double the moment when the principal vector of the forces for a rod is equal to zero. In Fig. 6, for $M \geqslant 0$ we show sections of the set Ω^* for Zhukovskii's problem by the planes $F_x = \text{const}$; the values of F_x are shown in the figure. In Fig. 7, for $F_y \geqslant 0$, we show sections of the set Ω^* by the planes $M = \text{const}$; the values of M are indicated in the figure. In view of the symmetry of the set Ω^* about the planes $M = 0$

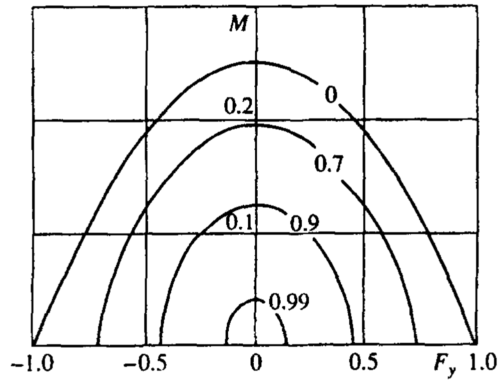

Fig. 3

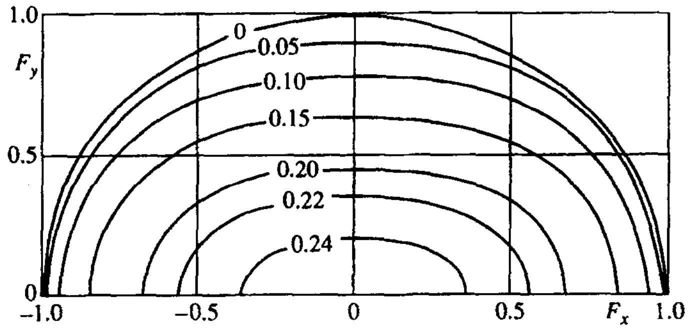

Fig. 4

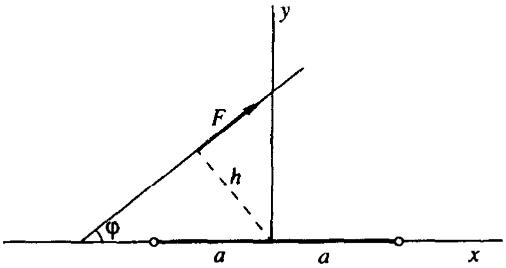

Fig. 5

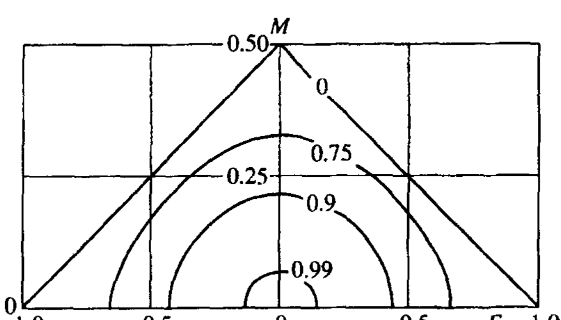

Fig. 6

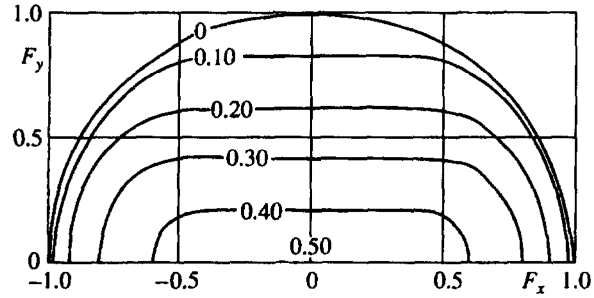

Fig. 7

and $F_y = 0$, in Figs 6 and 7, as in Figs 3 and 4, we only show half the corresponding sections. Notice both the general similarity and the considerable quantitative differences between the data shown in Figs 6 and 3 and Figs 7 and 4, respectively.

This research was supported financially by the Russian Foundation for Basic Research (02-01-00201).

REFERENCES

1. ZHUKOVSKII, N. Ye., The equilibrium conditions of a rigid body resting on a fixed plane with a certain area and able to move along this plane with friction. In *Collected Papers*. Gostekhteorizdat, Moscow, Leningrad, 1949, 1, 339–354.
2. CHERNOUSKO, F. L., The equilibrium conditions of a body on a rough plane. *Izv. Akad. Nauk SSSR. MTT*, 1988, 6, 6–17.

Translated by R.C.G.